AD-A013 688

LINEAR INVARIANT PREDICTION OF ORDER STATISTICS IN LOCATION AND SCALE FAMILIES

Kenneth S. Kaminsky, et al

Technology, Incorporated

Prepared for:

Aerospace Research Laboratories

June 1975

239088

ARL TR 75-0151

AD A013688

LINEAR INVARIANT PREDICTION OF ORDER STATISTICS IN LOCATION AND SCALE FAMILIES

TECHNOLOGY INCORPORATED
3821 COLONEL GLENN HIGHWAY
DAYTON, OHIO 45431

JUNE 1975

INTERIM REPORT 20 MAY 1974 — 16 AUGUST 1974

Approved for public release; distribution unlimited

APPLIED MATHEMATICS RESEARCH LABORATORY/LB
AEROSPACE RESEARCH LABORATORIES
Building 450 — Area B
Wright-Patterson Air Force Base, Ohio 45433

AIR FORCE SYSTEMS COMMAND
United States Air Force

NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Organizations or individuals receiving reports via Aerospace Research Laboratories automatic mailing lists should refer to the ARL number of the report received when corresponding about change of address or cancellation. Such changes should be directed to the specific laboratory orginating the report. Do not return this copy; retain or destroy.

Reports are not stocked by the Aerospace Research Laboratories. Copies may be obtained from:

National Technical Information Services
Clearinghouse
Springfield, VA 22161

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

ACCESSION for
NTIS White Section ☐
DDC Buff Section ☐
UNANNOUNCED ☐
JUSTIFICATION
DISTRIBUTION/AVAILABILITY CODES
Dist. AVAIL. and/or SPECIAL
A

Elizabeth Day

ELIZABETH DAY
Technical Documents
and STINFO Office

This report has been reviewed and cleared for open publication and public release by the appropriate Office of Information in accordance with AFR 190-12 and DODD 5230.9. There is no objection to unlimited distribution of this report to the public at large, or by DDC to the National Technical Information Service.

AIR FORCE/56780/15 July 1975 — 500

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARL 75-0151	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) LINEAR INVARIANT PREDICTION OF ORDER STATISTICS IN LOCATION AND SCALE FAMILIES		5. TYPE OF REPORT & PERIOD COVERED Technical - Interim 20 May 1974 - 16 Aug 1974
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Kenneth S. Kaminsky Nancy R. Mann Paul I. Nelson		8. CONTRACT OR GRANT NUMBER(s) F 33615-73-C-4155
9. PERFORMING ORGANIZATION NAME AND ADDRESS Technology Incorporated 3821 Colonel Glenn Highway Dayton, Ohio 45431		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DoD Element 61102F 7071-02-11
11. CONTROLLING OFFICE NAME AND ADDRESS Applied Mathematics Research Laboratory/LB Aerospace Research Laboratories Wright-Patterson AF Base, Ohio 45433		12. REPORT DATE JUNE 1975
		13. NUMBER OF PAGES 17
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Invariance; Linear estimate; Order statistic; Prediction; Reliability.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Identical items are simultaneously subjected to stress and operate independently until failure. We observe the early failures and use the times of these failure to predict the times of later failures in the same sample. Best linear invariant predictors are derived for location-scale families of failure time distributions, and two simplified predictors are developed. The relative behavior of the predictors is indicated in a table.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

PREFACE

The research for this report was performed by K.S. Kaminsky and P.I. Nelson of Bucknell University and by N.R. Mann of Rockwell International. The research of K.S. Kaminsky was supported under Contract F33615-73-C-4155, while he was a Technology Incorporated Visiting Research Associate at the Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio. That of N.R. Mann was supported under Contract F44620-71-C-0029. The period of the research was May 20, 1974 to August 17, 1974. The technical monitor of the project was Dr. H. Leon Harter, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

Some of the results in this report will appear in condensed form in *Biometrika* in late 1975.

TABLE OF CONTENTS

SECTION I

INTRODUCTION

Denote by $x_1 < x_2 < \cdots < x_n$ the times to failure of n items whose unordered failure times are independent and identically distributed. Assume that the underlying distribution of the unordered variates, $(1/\sigma)f\{(x-\mu)/\sigma\}$, is continuous and known up to location and scale. We consider the problem of predicting x_m after observing only $x_1, \ldots, x_r$, where $1 \leq r < m \leq n$. Prediction intervals in this setting have been studied by several authors (Hewett [1], Lawless [2], Kaminsky and Nelson [3], Likes [4], Mann, Schafer and Singpurwalla [5], Mann and Grubbs [6]). Best linear unbiased prediction of x_m was treated by Kaminsky and Nelson [7]. A significant reduction in mean square error is achieved by sacrificing unbiasedness and investigating the larger class of linear invariant predictors with mean square error proportional to σ^2. We find the best linear invariant predictors and we develop two simplified linear invariant predictors. The relative behavior of the predictors is indicated in Table 1 for the exponential, chi, normal, logistic, extreme value and double exponential distributions.

SECTION II

NOTATION

The order statistics can be written (Lloyd [8]) $x_i = \mu + \alpha_i\sigma + \varepsilon_i$ $(i=1, \ldots, n)$. We will write the first r of these in the matric form $X = A\Theta + \varepsilon$ where $X' = (x_1, \ldots, x_r)$, $A = (1,\alpha)$, $1' = (1, \ldots, 1)(1 \times r)$, $\alpha' = (\alpha_1, \ldots, \alpha_r)$, $\Theta' = (\mu,\sigma)$ and $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_r)$. Thus, our problem is to predict $x_m = A_m\Theta + \varepsilon_m$ $\{A_m = (1,\alpha_m)\}$ from X. The variance-covariance matrix of X is $\sigma^2 V = \sigma^2(v_{ij})(1 \leq i, j \leq r)$. Of course, α and V do not depend on μ or σ and $E(\varepsilon_i) = 0$ $(i=1, \ldots, n)$. Denote $\mathrm{cov}(X',x_m) = \sigma^2(v_{1m}, \ldots, v_{rm})$ by $\sigma^2 w'$.

When both μ and σ are unknown, we write $\hat{\Theta}' = (\hat{\mu},\hat{\sigma})$ and $\hat{x}_m$ for the best linear unbiased estimate of Θ' and best linear unbiased predictor of x_m, respectively. If one of the parameters is known, that parameter will appear as a subscript in the estimate and predictor. Thus, for example, $\hat{x}_{m\mu}$ and $\hat{\sigma}_\mu$ are respectively the best linear unbiased predictor of x_m and the best linear unbiased estimate of σ when μ is known. The best linear unbiased predictors and their mean square errors are given implicitly in Theorem 1.

SECTION III

BEST LINEAR INVARIANT PREDICTION OF x_m

There is a close connection between best linear invariant prediction and best linear invariant estimation, as we see in the following theorem. It should be pointed out that the theorem is not limited to the order statistics but could be stated for the general linear model.

Theorem 1. Let $k_1 = 1 - w'V^{-1}1$ and $k_2 = \alpha_m - w'V^{-1}\alpha$ and $k' = (k_1,k_2)$. Let the variance-covariance matrix of $(k'\hat{\theta},\hat{\sigma})$ be

$$\sigma^2\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$$

and $\mathrm{var}(\hat{\sigma}_\mu) = c_\mu\sigma^2$.

(a) If μ and σ are unknown, the best linear invariant predictor of x_m is

$$\begin{aligned}\tilde{x}_m &= w'V^{-1}X + k_1\hat{\mu} + \{k_2 - c_{12}/(1 + c_{22})\}\hat{\sigma} \\ &= \hat{x}_m - \{c_{12}/(1 + c_{22})\}\hat{\sigma}\end{aligned}$$

with mean square error

$$\begin{aligned}M(\tilde{x}_m) &= \sigma^2\{v_{mm} - w'V^{-1}w + c_{11} - c_{12}^2/(1 + c_{22})\} \\ &= M(\hat{x}_m) - c_{12}^2\sigma^2/(1 + c_{22}).\end{aligned}$$

(b) If μ is known and σ unknown, the best linear invariant predictor of x_m is

$$\begin{aligned}\tilde{x}_m &= w'V^{-1}X + k_1\mu + k_2\hat{\sigma}_\mu/(1 + c_\mu) \\ &= \hat{x}_{m\mu} + k_2c_\mu\hat{\sigma}_\mu/(1 + c_\mu)\end{aligned}$$

with mean square error

$$M(\tilde{x}_m) = \sigma^2\{v_{mm} - w'V^{-1}w + k_2^2c_\mu/(1 + c_\mu)\}$$

$$= M(\hat{x}_m) - k_2^2 c_\mu^2 \sigma^2/(1 + c_\mu).$$

(c) If σ is known and μ unknown, the best linear invariant predictor of x_m is

$$\tilde{x}_{m\sigma} = w'V^{-1}X + k_1\hat{\mu}_\sigma + k_2\sigma = \hat{x}_{m\sigma}$$

with mean square error

$$M(\tilde{x}_{m\sigma}) = \sigma^2(v_{mm} - w'V^{-1}w + c_{11} - c_{12}^2/c_{22}) = M(\hat{x}_{m\sigma}).$$

Proof. We prove part (a). We first point out that for any linear predictor $a'X$, the mean square error, $E(x_m - a'X)^2$ can be written as

$$M(a'X) = \sigma^2(v_{mm} - w'V^{-1}w) + E(p'X - k'\Theta)^2, \tag{1}$$

where $p = a - V^{-1}w$. This follows from the easily verified facts,

i) $x_m - a'X = \varepsilon_m - w'V^{-1}\varepsilon - (p'X - k'\Theta)$.

ii) $E(\varepsilon_m - w'V^{-1}\varepsilon)(p'X - k'\Theta) = 0$,

and

iii) $E(\varepsilon_m - w'V^{-1}\varepsilon)^2 = \sigma^2(v_{mm} - w'V^{-1}w)$.

Thus, in order that $a'X$ be the best linear invariant predictor of x_m, $p'X$ must be the best linear invariant estimate of $k'\Theta$. From Theorem 1 of Mann [9], we see that $p'X = k'\Theta + \{c_{12}/(1 + c_{22})\}\hat{\sigma}$ (i.e., $\tilde{x}_m = w'V^{-1}X + k_1\hat{\mu} + \{k_2 - c_{12}/(1 + c_{22})\}\hat{\sigma}$) and $E(p'X - k'\Theta)^2 = \sigma^2\{c_{11} - c_{12}^2/(1 + c_{22})\}$. That $\tilde{x}_m = \hat{x}_m - \{c_{12}/(1 + c_{22})\}\hat{\sigma}$ and $M(\tilde{x}_m) = M(\hat{x}_m) - c_{12}^2\sigma^2/(1 + c_{22})$ now follow from the results of Kaminsky and Nelson [7]. Parts (b) and (c) follow in a similar fashion from the results of Mann [10] and Kaminsky and Nelson [7].

Remark. The mean square error (1) can be rewritten in the more convenient form

$$M(a'X) = \sigma^2\{v_{mm} + a'Va - 2a'w + (\alpha_m - a'\alpha)^2\}$$
$$+ (1 - a'1)^2\mu^2 + 2(1 - a'1)(\alpha_m - a'\alpha)\mu\sigma.$$

Thus, $a'X$ is invariant for x_m if and only if $a'1 = 1$ and is unbiased for x_m if and only if, in addition, $a'\alpha = \alpha_m$.

Example 1. Assume that the parent population is exponential, $(1/\sigma)\exp\{-(x - \mu)/\sigma\}$, $x > \mu$, $\sigma > 0$, μ and σ unknown. We find $w'V^{-1} = (0,\dots,0,1)$; $\hat{\mu} = x_1 - \hat{\sigma}/n$; $\hat{\sigma} = -(n - 1)x_1/(r - 1) + \Sigma_{i=2}^{r-1} x_i/(r - 1) + (n - r + 1)x_r/(r - 1)$; $\mathrm{var}(\hat{\sigma}) = \sigma^2/(r - 1)$; $\mathrm{var}(\hat{\mu}) = \sigma^2/\{n^2(r - 1)\}$; $\mathrm{cov}(\hat{\mu},\hat{\sigma}) = -\sigma^2/\{n(r - 1)\}$; $k_1 = 0$; $k_2 = \delta_1$, where we define $\delta_i = \delta_i(r,m) = \Sigma_{j=r+1}^{m}(n - j + 1)^{-i}$ $(i=1,2)$. Now, $\tilde{x}_m = x_r + \delta_1\hat{\sigma}(r - 1)/r$ and $M(\tilde{x}_m) = \sigma^2(\delta_2 + \delta_1^2/r)$. For comparison, $\hat{x}_m = x_r + \delta_1\hat{\sigma}$ and $M(\hat{x}_m) = \sigma^2\{\delta_2 + \delta_1^2/(r-1)\}$.

SECTION IV

SIMPLIFIED LINEAR INVARIANT PREDICTION

Remark. In the interest of brevity, we will assume throughout this section that both μ and σ are unknown. Also, we will treat only invariant prediction (not necessarily unbiased). The corresponding unbiased predictors can be obtained in a manner quite similar to that outlined below by adding the appropriate linear constraint.

There are only a few distributions of which we know (the exponential, power-function, Pareto and a few others) where the best linear invariant predictor has a simple closed form. This is because V can rarely be inverted algebraicly. When r is large, inversion of V can be especially troublesome. These observations have motivated us to search for simplified predictors which do not depend on inversion of V. On examination of the coefficients of $\tilde{x}_m$ for various r, m and n, some patterns emerge. First, in distributions not having an unknown lower terminus, such as the normal, logistic, extreme value and double exponential, the first $r - 1$ coefficients are, in general, reasonably close to one another with a jump occurring at the r-*th* coefficient. Second, in distributions with an unknown lower terminus, such as the exponential and chi distributions, the middle $r - 2$ coefficients of $\tilde{x}_m$ are generally reasonably close to each other (with equality for the exponential distribution), and jumps occur at the first and last coefficients. Also, many researchers have observed that linear estimates of location and scale lose efficiency rather slowly as the coefficients are changed, while maintaining unbiasedness or invariance (cf. David [11], p. 108). These observations lead us to suggest using linear invariant predictors depending on two weights if the parent distribution does not have an unknown lower terminus and on three weights otherwise. Incidentally, Table 1 indicates that the two-weight predictors are fairly efficient even in distributions which *do* possess an unknown lower terminus.

Our two-weight linear invariant predictor takes the form

$$x_m^*(\lambda) = \lambda\Sigma_{i=1}^{r-1}x_i/(r - 1) + (1 - \lambda)x_r,$$

where λ is some real constant. The mean square error of $x_m^*(\lambda)$, viewed as a function of λ, is a parabola, opening upward, with its minimum value at

$$\lambda_o = \frac{v_{rr} - v_{rm} + \Sigma(v_{im} - v_{ir})/(r - 1) - \{\alpha_r - \Sigma\alpha_i/(r - 1)\}(\alpha_m - \alpha_r)}{v_{rr} + \Sigma\Sigma v_{ij}/(r - 1)^2 - 2\Sigma v_{ir}/(r - 1) + \{\alpha_r - \Sigma\alpha_i/(r - 1)\}^2},$$

all summations being from 1 to $r - 1$. We write x_m^* in place of $x_m^*(\lambda_o)$ and we observe that inversion of V is unnecessary to compute either x_m^* or its mean square error.

The three-weight linear invariant predictor takes the form

$$x_m^{**}(\beta,\gamma) = \beta x_1 + \gamma\Sigma_{i=2}^{r-1}x_i/(r - 2) + (1 - \beta - \gamma)x_r$$

The mean square error of $x_m^{**}(\beta,\gamma)$, viewed as a function of β and γ, is a paraboloid opening upward. It is elementary to verify that the minimum mean square error occurs at the point

$$\begin{pmatrix}\beta_o \\ \gamma_o\end{pmatrix} = \begin{pmatrix}\xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22}\end{pmatrix}^{-1}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}$$

where

$$\xi_{11} = v_{11} - 2v_{r1} + v_{rr} + (\alpha_r - \alpha_1)^2,$$

$$\xi_{12} = \Sigma(v_{i1} - v_{ir})/(r - 2) + v_{rr} - v_{r1} + (\alpha_r - \alpha_1)\{\alpha_r - \Sigma\alpha_i/(r - 2)\},$$

$$\xi_{22} = \Sigma\Sigma v_{ij}/(r - 2)^2 - 2\Sigma v_{ir}/(r - 2) + v_{rr} + \{\alpha_r - \Sigma\alpha_i/(r - 2)\}^2,$$

$$\eta_1 = v_{1m} - v_{rm} - v_{1r} + v_{rr} - (\alpha_r - \alpha_1)(\alpha_m - \alpha_r),$$

and

$$\eta_2 = \Sigma(v_{im} - v_{ir})/(r - 2) - v_{rm} + v_{rr} - (\alpha_m - \alpha_r)\{\alpha_r - \Sigma\alpha_i/(r - 2)\},$$

where all summations run from 2 to $r - 1$. We write x_m^{**} in place of $x^{**}(\beta_0,\gamma_0)$ and we note again that calculation of x_m^{**} or its mean square error does not require inversion of V.

Remark. In the case of the exponential distribution, $\tilde{x}_m$ is a three-weight predictor (see Example 1), so that $\tilde{x}_m \equiv x_m^{**}$.

SECTION V

DISCUSSION OF THE TABLE

The computations for Table 1 were performed on the Cyber 74 computer at the Wright-Patterson Air Force Base, Ohio. The arithmetic was accurate to fourteen decimal places but was, of course, limited by the number of significant figures in the tables of expectations and covariances. These expectations and covariances were obtained from the following sources: Govindarajulu and Eisenstat [12], the chi distribution

$$f(x) = \sqrt{2/\pi}e^{-x^2/2}, \quad x > 0;$$

Sarhan and Greenberg [13], pp. 190-205, the normal distribution

$$f(x) = e^{-x^2/2}/\sqrt{2\pi}, \quad -\infty < x < \infty,$$

Birnbaum and Dudman [14], Gupta and Shah [15], Tartar and Clark [16], Shah [17] and Gupta, Qureishi and Shah [18], the logistic distribution

$$F(x) = \frac{\pi}{\sqrt{3}} \frac{e^{-\pi x/\sqrt{3}}}{(1 + e^{-\pi x/\sqrt{3}})^2}, \quad -\infty < x < \infty;$$

Mann [19], the extreme value distribution

$$f(x) = e^{x-e^x}, -\infty < x < \infty;$$

and Govindarajulu [20], the double exponential distribution

$$f(x) = e^{-|x|}/2, \quad -\infty < x < \infty.$$

Table 1 is intended to give some indication of the relative performance of the predictors in small and moderately large samples. The table contains the mean square errors of $\hat{x}_m$, $\tilde{x}_m$, x_m^* and x_m^{**} (μ and σ unknown) for n = 7, n = 20 and various combinations of r and m, for the five distributions mentioned above and the exponential distribution. In most cases, the best linear invariant predictor is seen to reduce mean square error considerably below that of the best linear unbiased predictor. The three-weight predictor generally comes quite close to the best linear invariant predictor, and, in most cases, both simplified predictors surpass the best linear unbiased predictor. It seems likely from the table that the discrepancy between the mean square errors of the unbiased predictor and the invariant predictors increases the farther away m is from r. We will illustrate some of the computations with an example.

Example 2. Seven identical items whose failure times are known to follow the chi distribution with unknown location and scale are simultaneously subjected to stress. The items function independently, and, after the first four failures have been observed, we wish to predict the time of the last failure. From Govindarajulu and Eisenstat [12], we find $\hat{\mu} = 1.3108x_1 - 0.0106x_2 - 0.0246x_3 - 0.2753x_4$ and $\hat{\sigma} = -2.0147x_1 + 0.1370x_2 + 0.2093x_3 + 1.6685x_4$, $\mathrm{var}(\hat{\mu}) = 0.0304\sigma^2$, $\mathrm{var}(\hat{\sigma}) = 0.2388\sigma^2$ and $\mathrm{cov}(\hat{\mu},\hat{\sigma}) = -0.0485\sigma^2$. Also, we find that $w'V^{-1} = (-0.0011, -0.0018, -0.0037, 0.6443)$, $w'V^{-1}1 = 0.6377$ and $w'V^{-1}\alpha = 0.4497$. The four predictors are $\hat{x}_7 = -2.0934x_1 + 0.1693x_2 + 0.2534x_3 + 2.6707x_4$, $\tilde{x}_7 = -1.6270x_1 + 0.1375x_2 + 0.2051x_3 + 2.2844x_4$, $x_7^* = -0.6694(x_1 + x_2 + x_3) + 3.0082x_4$ and $x_7^{**} = -1.6401x_1 + 0.1732(x_2 + x_3) + 2.2937x_4$. From Table 1 with $r = 4$, $m = 7$ and $n = 7$ for the chi distribution, we find the mean square errors of the predictors to be $M(\hat{x}_7) = 0.5998\sigma^2$, $M(\tilde{x}_7) = 0.5335\sigma^2$, $M(x_7^*) = 0.5772\sigma^2$ and $M(x_7^{**}) = 0.5335\sigma^2$.

Table I. *Mean square errors of the predictors divided by* σ^2.

	$M(\hat{x}_m)$	$M(\tilde{x}_m)$	$M(x_m^*)$	$M(x_m^{**})$	$M(\hat{x}_m)$	$M(\tilde{x}_m)$	$M(x_m^*)$	$M(x_m^{**})$
	$r = 4,\ m = 5,\ n = 7$				$r = 4,\ m = 6,\ n = 7$			
Exponential	0.1482	0.1389	0.1447	0.1389	0.5926	0.5347	0.5709	0.5347
Chi	0.0644	0.0606	0.0628	0.0606	0.2066	0.1876	0.1994	0.1876
Normal	0.1379	0.1308	0.1332	0.1309	0.3754	0.3449	0.3554	0.3453
Logistic	0.1178	0.1109	0.1117	0.1110	0.3478	0.3165	0.3200	0.3170
Extreme value	0.1756	0.1643	0.1647	0.1644	0.4225	0.3783	0.3797	0.3786
Double exponential	0.1818	0.1702	0.1702	0.1702	0.6456	0.5830	0.5832	0.5831
	$r = 4,\ m = 7,\ n = 7$				$r = 5,\ m = 6,\ n = 20$			
Exponential	2.4815	2.2014	2.3764	2.2014	0.0056	0.0053	0.0055	0.0053
Chi	0.5998	0.5335	0.5772	0.5335	0.0050	0.0048	0.0049	0.0048
Normal	0.9147	0.8239	0.8555	0.8251	0.0275	0.0266	0.0268	0.0266
Logistic	1.0507	0.9437	0.9542	0.9454	0.0255	0.0244	0.0245	0.0245
Extreme value	0.8453	0.7325	0.7355	0.7331	0.0646	0.0620	0.0621	0.0621
Double exponential	2.5376	2.2605	2.2611	2.2609	0.0501	0.0481	0.0481	0.0481
	$r = 5,\ m = 13,\ n = 20$				$r = 5,\ m = 20,\ n = 20$			
Exponential	0.2002	0.1739	0.1938	0.1739	4.3331	3.7826	4.2002	3.7826
Chi	0.1210	0.1058	0.1171	0.1058	1.0484	0.9066	1.0219	0.9066
Normal	0.3438	0.3065	0.3153	0.3070	1.7008	1.4939	1.5427	1.4964
Logistic	0.3111	0.2707	0.2715	0.2709	2.1589	1.8702	1.8741	1.8714
Extreme value	0.6362	0.5517	0.5521	0.5518	1.9460	1.6343	1.6354	1.6346
Double exponential	0.4976	0.4338	0.4338	0.4338	5.1414	4.4474	4.4474	4.4474
	$r = 10,\ m = 11,\ n = 20$				$r = 10,\ m = 15,\ n = 20$			
Exponential	0.0111	0.0110	0.0114	0.0110	0.1325	0.1278	0.1432	0.1278
Chi	0.0063	0.0063	0.0064	0.0063	0.0571	0.0552	0.0614	0.0553
Normal	0.0158	0.0157	0.0159	0.0157	0.1102	0.1072	0.1125	0.1081
Logistic	0.0126	0.0125	0.0125	0.0125	0.0940	0.0909	0.0920	0.0914
Extreme value	0.0217	0.0215	0.0216	0.0215	0.1316	0.1267	0.1274	0.1271
Double exponential	0.0148	0.0146	0.0146	0.0146	0.1421	0.1370	0.1371	0.1371
	$r = 10,\ m = 20,\ n = 20$				$r = 15,\ m = 16,\ n = 20$			
Exponential	2.5030	2.4077	2.7243	2.4077	0.0429	0.0427	0.0443	0.0427
Chi	0.4681	0.4498	0.5202	0.4499	0.0142	0.0142	0.0147	0.0142
Normal	0.6369	0.6142	0.6544	0.6211	0.0234	0.0233	0.0238	0.0234
Logistic	0.9086	0.8771	0.8838	0.8796	0.0215	0.0214	0.0217	0.0216
Extreme value	0.4876	0.4605	0.4635	0.4622	0.0198	0.0197	0.0198	0.0198
Double exponential	2.5189	2.4237	2.4239	2.4239	0.0428	0.0427	0.0429	0.0428
	$r = 15,\ m = 18,\ n = 20$				$r = 15,\ m = 20,\ n = 20$			
Exponential	0.2574	0.2545	0.2800	0.2545	1.8360	1.8112	2.0272	1.8112
Chi	0.0675	0.0669	0.0734	0.0669	0.2899	0.2863	0.3253	0.2865
Normal	0.0993	0.0984	0.1043	0.1000	0.3685	0.3642	0.3935	0.3722
Logistic	0.1051	0.1042	0.1077	0.1061	0.5988	0.5928	0.6128	0.6044
Extreme value	0.0741	0.0731	0.0743	0.0739	0.2041	0.2001	0.2045	0.2030
Double exponential	0.2564	0.2536	0.2567	0.2559	1.8266	1.8030	1.8293	1.8224

SECTION VI

SOME COMMENTS ON TWO-SAMPLE PREDICTION

The procedures discussed in the preceding sections extend easily to two-sample prediction. Specifically, suppose we wish to predict the m'-*th* failure time, $y_{m'}$, in a future, independent sample of size n' from the same population, the predictor being based on X, where now r may equal n. Prediction intervals in this setting have been studied by Mann and Saunders [9], Lawless [21, 22, 23], Antle and Rademaker [24], Kaminsky and Nelson [3], Mann, Schafer and Singpurwalla [5] and Fertig and Mann [25].

It is easy to show that $\hat{y}_{m'} = \hat{E}(y_{m'})$ and $\tilde{y}_{m'} = \tilde{E}(y_{m'})$. That is, the best predictor of $y_{m'}$ is the best estimate of $E(y_{m'})$ in both the unbiased and the invariant cases. This follows from the independence of the two samples. Simplified predictors of $y_{m'}$ can also be derived as in §4. We will not pursue these ideas further at this time.

REFERENCES

[1] Hewett, J.E., "A Note on Prediction Intervals Based on Partial Observations in Certain Life Test Experiments," *Technometrics*, 10, 850-3, 1968.

[2] Lawless, J.F., "A Prediction Problem Concerning Samples from the Exponential Distribution with Applications in Life Testing," *Technometrics*, 13, 725-30, 1971.

[3] Kaminsky, K.S. & Nelson, P.I., "Prediction Intervals for the Exponential Distribution Using Subsets of the Data," *Technometrics*, 16, 57-9, 1974.

[4] Likeš, J., "Prediction of sth Ordered Observation for the Two-Parameter Exponential Distribution," *Technometrics*, 16, 241-4, 1974.

[5] Mann, N.R., Schafer, R.E. & Singpurwalla, N.D., *Methods for Statistical Analysis of Reliability and Life Data*, New York: Wiley, 1974.

[6] Mann, N.R. & Grubbs, F.E., "Simple, Efficient Closed-Form Approximations for Beta Percentiles, Exponential Prediction Intervals and Confidence Bounds on Exponential and Binomial Parameters," *J. Amer. Statist. Assoc.*, 69, 654-61, 1974.

[7] Kaminsky, K.S. & Nelson, P.I., "Best Linear Unbiased Prediction of Order Statistics in Location and Scale Families," *J. Amer. Statist. Assoc.*, 70, to appear in 1975.

[8] Lloyd, E.H., "Least-Squares Estimation of Location and Scale Parameters Using Order Statistics," *Biometrika*, 39, 88-95, 1952.

[9] Mann, N.R. & Saunders, S.C., "On Evaluation of Warranty Assurance when Life has a Weibull Distribution," *Biometrika*, 56, 615-25, 1969.

[10] Mann, N.R., "Optimum Estimators for Linear Functions of Location and Scale Parameters," *Ann. Math. Statist.*, 40, 2149-55, 1969.

[11] David, H.A., *Order Statistics*, New York: Wiley, 1970.

[12] Govindarajulu, Z. & Eisenstat, S., "Best Estimates of Location and Scale Parameters of a Chi (1 D.F.) Distribution, Using Ordered Observations," *Rep. Statist. Appl. Res. JUSE*, 12, 149-64, 1965.

[13] Sarhan, A.E. & Greenberg, B.G. (eds.), *Contributions to Order Statistics*, New York: Wiley, 1962.

[14] Birnbaum, A. & Dudman, J., "Logistic Order Statistics," *Ann. Math. Statist.*, 34, 658-63, 1963.

[15] Gupta, S.S. & Shah, B.K., "Exact Moments and Percentage Points of the Order Statistics and the Distribution of the Range from the Logistic Distribution, *Ann. Math. Statist.*, 36, 907-20, 1965.

[16] Tartar, M.E. & Clark, V.A., "Properties of the Median and Other Order Statistics of Logistic Variates," *Ann. Math. Statist.*, 36, 1779-86, 1965.

[17] Shah, B.K., "On the Bivariate Moments of Order Statistics from a Logistic Distribution," *Ann. Math. Statist.*, 37, 1002-10, 1966.

[18] Gupta, S.S., Qureishi, A.S. & Shah, B.K., "Best Linear Unbiased Estimators of the Parameters of the Logistic Distribution Using Order Statistics," *Technometrics*, 9, 43-56, 1967.

[19] Mann, N.R., "Results on Statistical Estimation and Hypothesis Testing with Applications to the Weibull and Extreme-Value Distributions," Aerospace Research Laboratories Report ARL 68-0068, AD 672 979, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, 1968.

[20] Govindarajulu, Z., "Best Linear Estimates Under Symmetric Censoring of the Parameters of a Double Exponential Population," *J. Amer. Statist. Assoc.*, 61, 248-58, 1966.

[21] Lawless, J.F., "Two Sample Prediction Results for the Exponential Distributions, with Tables and Applications," University of Manitoba Technical Report No. 9, 1971.

[22] Lawless, J.F., "On the Estimation of Safe Life When the Underlying Life Distribution is Weibull," *Technometrics*, 15, 857-66, 1973.

[23] Lawless, J.F., "Approximations to Confidence Intervals for Parameters in the Extreme Value and Weibull Distributions," *Biometrika*, 61, 123-9, 1974.

[24] Antle, C.E. & Rademaker, F., "An Upper Limit on the Maximum of m Future Observations from a Type 1 Extreme Value Distribution," *Biometrika*, 59, 475-7, 1972.

[25] Fertig, K.W. & Mann, N.R., "A New Approach to the Determination of One-Sided Prediction Intervals for Normal and Lognormal Distributions, with Tables," to appear in the proceedings of the Conference on Reliability and Fault-Tree Analysis, University of California, Berkeley, 1974. To be published by the Society for Industrial and Applied Mathematics, Philadelphia.

*U.S.Government Printing Office: 1975 — 657-023/677